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The use of asymptotic modelling in vibration and stability analysis of structures

S. Ilanko*

Department of Mechanical Engineering, University of Canterbury, Christchurch, New Zealand

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Abstract

The use of springs with very large stiffness to model constraints in vibratory systems has been a popular approach to overcome the limitations on the choice of admissible functions in the Rayleigh–Ritz method. The maximum possible error resulting from this asymptotic modelling can be determined by using positive and negative stiffness values, or in general terms using positive and negative penalty functions. This paper illustrates how this method could be used to determine the critical loads of structures.

1. Introduction

The Rayleigh–Ritz method is a well-known technique for solving vibration, stability and stress analysis problems. In his address to the American Mathematical Society, Courant [1] identifies the difficulty in choosing admissible functions as the major drawback of this method. This difficulty arises because of the need to satisfy the geometric constraints of a system. To overcome this, he suggests the use of artificial stiffness parameters having very large magnitude so that a rigid constraint could be approximately modelled by a restraint. The effectiveness of this approach in determining natural frequencies has since been studied by several researchers, and its applicability has been extended to analyze rigidly connected systems and systems with cracks [2–7]. This idea has also been used extensively in the solution of constrained variational equations and optimization problems through the use of penalty functions [8–12]. In solving such problems, the individual trial functions are allowed to violate the essential conditions (constraint conditions). These conditions are indirectly imposed on the sum of the trial functions by adding an error

*Tel.: +3-3366-7001; fax: +3-3364-2078.
E-mail address: ilanko@mech.canterbury.ac.nz (S. Ilanko).

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function to the functional to be minimized. This error function is a product of the square of the constraint condition and a very large penalty parameter.

The main obstacle to this approach is that there is no easy way to determine a suitable value for the penalty parameter. While the parameter must be large enough to effect a constraint, it must be small enough to avoid any numerical problems [10,12]. The limitations of the latter depend on the capacity of the computer and the algorithm used, and must be taken into consideration when using any penalty function. As Zienkiewicz [10] has pointed out, the selection of penalty parameter is done empirically. However, the author is hopeful that a solution to this problem can be found. In two recent publications [13,14], it has been shown that the use of positive and negative values for the stiffness of the artificial restraints gives natural frequencies, which bracket the corresponding values for a rigidly constrained system. This means that the maximum error introduced by the asymptotic modelling (i.e., replacing rigid constraints with restraints having large stiffness—equivalent to the use of large penalty parameters) can be found by calculating the frequencies of the system using artificial restraints with positive and negative stiffness values. Since the results for the rigidly constrained systems would be bounded by these results, if the difference in the frequencies calculated using positive and negative values were too large then the magnitude of stiffness could be increased. Starting with reasonably large stiffness parameters, this process could be repeated until the desired accuracy is achieved. In a recent publication [15], the author has shown that the constrained solution for the deflection of a beam is also bounded by results from mathematical models using positive and negative penalty parameters. If this is true in general, any error due to the use of penalty functions in solving a variational problem may be determined, and controlled, by using positive and negative values for the penalty parameter.

The purpose of this paper is to show how this approach can be used to determine the critical loads of a constrained structure in a linear analysis. It is shown that the critical loads obtained by using positive and negative values for the stiffness of artificial restraints bracket the critical loads of the constrained structure. It is also shown that as the magnitudes of the stiffness of the restraints approach infinity, the critical loads of the restrained structure would approach the critical loads of the constrained structure. Results for a clamped-simply supported beam are presented here to illustrate this.

2. Theoretical considerations

Consider a structure A subject to a given set of static loads $\gamma \boldsymbol{P}$, where γ is a load factor and $\boldsymbol{P}$ represents forces or moments applied to the structure. In a typical stability problem, the ratio of the various forces and moments remain the same and the problem is that of finding the critical load factor which causes instability. Let A_r be a modified structure which is obtained by adding r restraints to A. This may be physically achieved by using springs of stiffness k. In general, this stiffness coefficient may be positive or negative. Let $\tilde{A}_r$ be the corresponding structure where the restraints are replaced by rigid constraints.

The linear stability problem may be expressed by an eigenvalue equation of the form

$$[\mathbf{K}]\{\mathbf{q}\} - \gamma[\mathbf{S}]\{\mathbf{q}\} = \{\mathbf{0}\}. \tag{1}$$

Here [**K**] is an elastic stiffness matrix, [**S**] is a geometric stiffness matrix, and {**q**} is a generalized displacement vector. The roots of the above eigenvalue equation give the critical values of γ for which non-zero {**q**} is possible.

Now consider this as a special case of the vibration problem expressed as an eigenvalue equation in terms of a mass matrix [**M**] in the form

$$[\mathbf{K}]\{\mathbf{q}\} - \gamma[\mathbf{S}]\{\mathbf{q}\} - \omega^2[\mathbf{M}]\{\mathbf{q}\} = \{\mathbf{0}\}, \tag{2}$$

where ω the natural frequency. For the special case of $\omega = 0$, this reduces to Eq. (1). All real structures possess mass, but even if a structure had negligible mass one could attach an artificial mass distribution for the purpose of this proof. Now consider the possibility of a vibratory motion of the structure with actual or artificial mass(es) in the neighbourhood of the critical values of γ. At a critical equilibrium, Eq. (1) holds. Therefore the corresponding natural frequency must be zero, as otherwise Eq. (2) cannot be satisfied. A physical explanation for this statement is described below.

As a natural frequency of a structure approaches zero, the period of vibration approaches infinity. This means, if the structure were given an initial displacement from its equilibrium state, it will never return to its original state, which by definition indicates a critical equilibrium state. This argument applies to linear analysis only, and it should be noted that there are non-linear cases (bifurcation of shells) where instability may occur in the absence of a corresponding zero natural frequency.

From Rayleigh's theorem of separation [16,17], the nth natural frequency of a constrained structure is bracketed by the nth and $(n+1)$th natural frequencies of the unconstrained structure. The unconstrained structure may have partial elastic restraints. Applying this to systems A_1 and $\tilde{A}_1$, gives

$$\omega_{n,1} \leqslant \tilde{\omega}_{n,1} \leqslant \omega_{n+1,1}, \tag{3a}$$

where $\omega_{n,r}, \tilde{\omega}_{n,r}$ denote the nth natural frequency of systems A_r and $\tilde{A}_r$, respectively.

Therefore,

$$\omega^2_{n,1} \leqslant \tilde{\omega}^2_{n,1} \leqslant \omega^2_{n+1,1}. \tag{3b}$$

The above equation is valid even if the square of the frequencies were negative and whether or not the restraints have positive or negative stiffness values.

Let $\tilde{\gamma}_{n,1}$ be the nth critical load ratio of the constrained structure $\tilde{A}_1$.

If

$$\gamma = \tilde{\gamma}_{n,1} \quad \text{then} \;\; \tilde{\omega}_{n,1} = 0. \tag{4}$$

It follows from Eqs. (3b) and (4) that for the restrained structure A_1, for any positive, definite k, if

$$\gamma = \tilde{\gamma}_{n,1}, \quad \omega^2_{n,1} \leqslant 0. \tag{5}$$

Eq. (5) is true for any definite k, but interest is only in the case of $k > 0$.

Let $\gamma_{n,1}$ be the nth critical load ratio of A_1 for the same positive stiffness value. Noting that a critical state corresponds to a zero natural frequency, if

$$\gamma = \gamma_{n,1} \quad \text{then} \;\; \omega^2_{n,1} = 0. \tag{6}$$

One can define the load ratio in such a way that an increase in its value would cause the system to approach the critical states of the modes of interest. With this definition, it can be said that an increase in the load ratio γ cannot increase ω^2. Then it follows from Eqs. (5) and (6) that

$$\gamma_{n,1} \leqslant \tilde{\gamma}_{n,1}. \tag{7}$$

Similarly from Eqs. (3b) and (4) for the restrained structure A_1, for any negative, definite k, if

$$\gamma = \tilde{\gamma}_{n,1}, \quad \omega^2_{n+1,1} \geqslant 0. \tag{8}$$

Let $\gamma_{n+1,1}$ be the value of γ, which, for the same negative stiffness k, makes the $(n+1)$th frequency squared become zero. Thus $\gamma_{n+1,1}$ is the $(n+1)$th critical load of A_1 with a negative stiffness value.

If

$$\gamma = \gamma_{n+1,1} \quad \text{then } \omega^2_{n+1,1} = 0. \tag{9}$$

Since an increase in the load ratio γ cannot increase ω^2, from Eqs. (8) and (9)

$$\gamma_{n+1,1} \geqslant \tilde{\gamma}_{n,1}. \tag{10}$$

Combining Eqs. (7) and (10) gives

$$\gamma_{n+1,1} \geqslant \tilde{\gamma}_{n,1} \geqslant \gamma_{n,1}. \tag{11}$$

This may be regarded as a statement of separation of critical loads.

According to the theorems of existence and convergence of natural frequencies with negatively restrained structures [14]

$$\omega_{n,1} \to \tilde{\omega}_{n,1} \quad \text{as } k \to \infty, \tag{12a}$$

and

$$\omega_{n+1,1} \to \tilde{\omega}_{n,1} \quad \text{as } k \to -\infty. \tag{12b}$$

Using Eqs. (6) and (12a), as $k \to +\infty$, for $\gamma = \gamma_{n,1}, \tilde{\omega}_{n,1} \to 0$.

From this and Eq. (4), as

$$k \to +\infty, \quad \gamma_{n,1} \to \tilde{\gamma}_{n,1}. \tag{13a}$$

Similarly from Eqs. (9) and (12b), as $k \to -\infty$, for $\gamma = \gamma_{n+1,1}, \quad \tilde{\omega}_{n,1} \to 0$.

From this and Eq. (4), as $k \to -\infty$,

$$\gamma_{n+1,1} \to \tilde{\gamma}_{n,1}. \tag{13b}$$

From Eqs. (11) and (13) one can therefore say that the critical loads of a structure with one restraint would approach the critical loads of the same structure with a rigid constraint (corresponding to the restraint) as the magnitude of the stiffness parameter approaches infinity. Since the results for constrained systems would be bracketed by the results for a restrained system, the maximum possible error due the approximation in modelling the constraint with a restraint is the difference between the critical load ratios corresponding to positive and negative restraints.

By using the principle of mathematical induction, it may be shown that this is true for any number of restraints. This finding is useful in applying the Rayleigh–Ritz procedure for calculating the critical loads of constrained structures, since the individual deflection functions need not satisfy the constraint conditions. This is illustrated in the example presented in the next section.

The use of negative stiffness solves another problem. While the buckling loads obtained using large positive restraints in place of rigid constraints and connections are upper bounds to the asymptotic model they cannot be guaranteed to be upper-bound estimates of the actual problem because the asymptotic model is more flexible than the actual structure. Therefore, one gets a lower-bound estimate of an upper-bound solution resulting in an uncertainty about the nature of the boundedness. However, if negative values are used for the spring stiffness, then the buckling loads obtained cannot in any event be lower than the buckling loads of the fully constrained structure. This way one can get a true upper-bound solution.

3. Illustrative example

In order to illustrate the behaviour of a system with a support approximated by a spring of positive and negative stiffness, consider an Euler–Bernoulli beam of length L, flexural rigidity EI, clamped at one end and laterally supported at the other by a spring of stiffness k' (ultimately to approximate a simple support) which is subject to a compressive axial load P. The exact solution for this problem is readily obtainable from the beam differential equation with substitution of the appropriate boundary conditions.

A Rayleigh–Ritz solution for the clamped–spring supported problem may be formulated as follows. The lateral displacement of the beam may be expressed by a power series

$$f(x) = \sum_{j=1}^{n} a_j (x/L)^{j+1}, \tag{14}$$

where n is the number of terms used. These functions satisfy the zero slope and displacement conditions at the clamped end ($x = 0$) but permit deflection to exist at the spring supported end ($x = L$).

The Rayleigh–Ritz minimization equation is

$$\partial V / \partial a_i = 0, \tag{15}$$

where the total potential energy V is given by

$$V = \int_0^L \frac{EI}{2} (f''(x))^2 \, \mathrm{d}x - \int_0^L \frac{P}{2} (f'(x))^2 \, \mathrm{d}x + \frac{k'}{2} (f(L))^2. \tag{16}$$

Substituting Eq. (14) into Eq. (16) and minimizing as given by Eq. (15) results in a matrix eigenvalue equation of the standard form which can be solved using any one of a number of standard algorithms. As mentioned before, the clamped-simply supported case is approached by letting the modulus of k' become very large. The results are given in Fig. 1 and Table 1, in terms of the critical load ratio $\gamma_{n,1} = P_{c,n,1} L^2 / EI$ and a non-dimensional stiffness coefficient $k = k' L^3 / EI$.

It may be seen that as the absolute value of the spring stiffness parameter is increased, the values of the critical load ratio converge toward those for a clamped-simply supported beam, approaching from below for the positive spring stiffness and from above for the negative stiffness, as predicted by Eqs. (11) and (13).

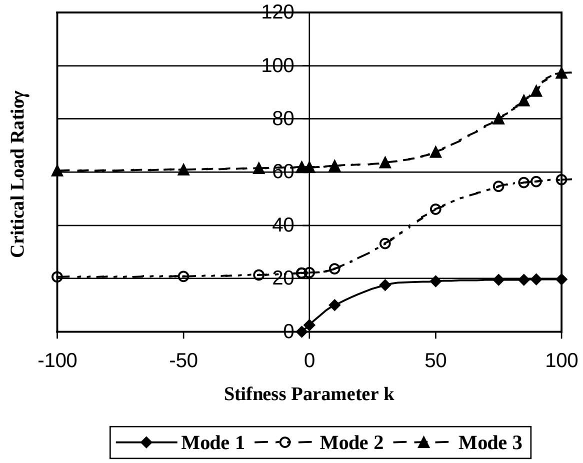

Fig. 1. Variation of the critical load with stiffness.

Table 1
Variation of the critical load with stiffness ($n = 8$)

Stiffness k	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{3,1}$
-10^5		20.191	59.681
-10^4		20.195	59.692
-10^3		20.230	59.793
-100		20.533	60.436
-50		20.781	60.781
-20		21.229	61.193
-3		21.966	61.593
-2.9995	0.00042	21.966	61.593
0	2.4674	22.207	61.686
10	9.956	23.640	62.070
30	17.558	33.073	63.460
50	18.992	45.988	67.168
75	19.496	54.637	79.620
85	19.597	55.966	86.272
90	19.637	56.419	89.620
100	19.703	57.076	96.027
10^3	20.150	59.554	118.816
10^4	20.187	59.668	119.063
10^5	20.190	59.679	119.084
$\pm\infty$ (exact)	20.191	59.68	

It is interesting to note that as k tends towards -3, where the absolute value of the stiffness becomes equal to that of a tip-loaded cantilever, the fundamental critical load parameter tends towards zero. This is to be expected because for $k = -3$, the structure with the negative restraint has no net stiffness even at the unloaded state as the stiffness of the original structure and the negative stiffness of the artificial restraint cancel each other. Therefore, when using negative stiffness in asymptotic modelling, it is best to calculate the stiffness of the unrestrained structure corresponding to the desired constraint, and use values that are larger in magnitude (in the present case the magnitude of negative stiffness should be greater than 3) to ensure that only the critical loads of the constrained structure are being delimited.

For $k = 0$, the critical load ratios corresponding to the first three modes (2.4674, 22.2069, 61.8626) agree very closely with the exact values of $\pi^2/4$, $3\pi^2/4$, and $5\pi^2/4$. The discrepancy between these values is less than 1%. As expected the critical loads calculated using the Rayleigh–Ritz method are slightly greater than the exact values. For this case, the constraint conditions (zero slope and zero deflection at the clamp) are satisfied. The exact values of critical load ratios for a clamped-simply supported beam are given in the last row. It may be seen that for very large magnitudes of k, the critical load ratios of the restrained system approach the exact values of the rigidly constrained system; for negative stiffness values the approach is from above while for positive stiffness it is from below.

4. Concluding remarks

In the determination of the critical loads of linear elastic structures, rigid constraints and connections may be modelled by elastic restraints having very large positive or negative stiffness coefficients. This is particularly useful in the application of the Rayleigh–Ritz method as it removes the limitation in selecting admissible functions. It is possible to select functions which violate boundary conditions and compatibility conditions at connections, by enforcing these constraints through the introduction of artificial stiffness parameters of very large positive and negative values. The critical loads found by using positive and negative stiffness approach the critical loads of rigidly constrained structures from different sides. Therefore, any error introduced by the asymptotic modelling could be estimated and in most cases controlled by choosing stiffness coefficients of sufficiently large magnitude.

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Message to Professor Arthur W. Leissa

It is a privilege to contribute to this special edition of the Journal of Sound and Vibration honouring Professor Arthur W. Leissa. My research career began in the late 1970s at Manchester University (UK) with a review of plate vibration literature and Professor Leissa's book on vibration of plates was a key starting point. Since then I have read many of Professor Leissa's books and articles which are both comprehensive and very well written making it easy even for researchers from different backgrounds to follow. In addition to his well-known and well-cited contribution to the vibration literature, I would also like to thank him for organizing the excellent symposia on vibration of continuous systems. I enjoyed the technical papers and the mountains, and look forward to taking part in future ones. I admire Professor Leissa's energy and enthusiasm for research, the passion to get to the bottom of everything in vibration and his ability to reach the top of both vibration research and the mountains. I wish Professor Arthur Leissa and Mrs. Trudi Leissa, a happy, adventurous and pleasant retirement and look forward to more of Professor Leissa's books and articles.

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